Graph Theory

Mongi BLEL

King Saud University

August 30, 2019

Table of contents

Definition

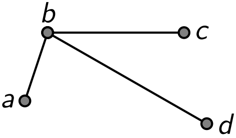
A graph (رسم) $G = (V, E)$ is a structure consisting of a set V of vertices (رؤُوس) (also called nodes), and a set E of edges (أضلَاع) , which are lines joining vertices.
Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints. If the edge e links the vertex a to the vertex b, we write $e = \{a, b\}$.
The order of a graph $G = (V, E)$ is the cardinality of its vertex set, and the size of a graph is the cardinality of its edge set.

There is several type of graphs, (undirected, directed, simple, multigraph,...) have different formal definitions, depending on what kinds of edges are allowed.

Definition

1. A simple graph (رسم بسيط) G is a graph that has no loops (عروَات), (that is no edge $\{a, b\}$ with $a = b$) and no parallel edges between any pair of vertices.
2. A multigraph G is a graph that has no loop and at least two parallel edges between some pair of vertices.

Simple Undirected Graph (رسم بسيط غير موجه)



Only undirected edges, at most one edge between any pair of distinct nodes, and no loops.

Directed Graph (Digraph) (with loops)

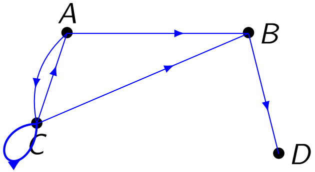
Definition

A directed graph (digraph) , $G = (V, E)$, consists of a non-empty set, V, of vertices (or nodes), and a set $E \subset V \times V$ of directed edges (or ordered pairs). Each directed edge $(a, b) \in E$ has a start (tail) vertex a, and a end (head) vertex b.

a is called the initial vertex (اَلرَّأس اَلإِبتدَائِي) and b is the terminal vertex (اَلرَّأس اَلنهَائِي).

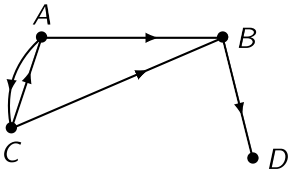
Note: a directed graph $G = (V, E)$ is simply a set V together with a binary relation E on V.

Example



Only directed edges, at most one directed edge from any node to any node, and loops are allowed.

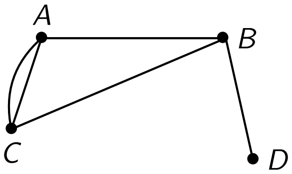
Simple Directed Graph



Only directed edges, at most one directed edge from any node to any other node, and no loops allowed.

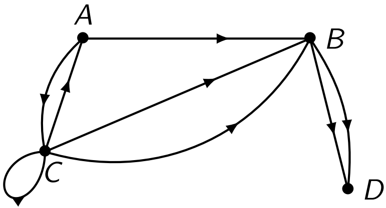
Definition

A (simple,undirected) multigraph, $G = (V, E)$, consists of a non-empty set V of vertices (or nodes), and a set $E \subset [V]^2$ of (undirected) edges, but no loops.



Only undirected edges, may contain multiple edges between a pair of nodes, but no loops.

Directed Multigraph:



Only directed edges, may contain multiple edges from one node to another, the loops are allowed.

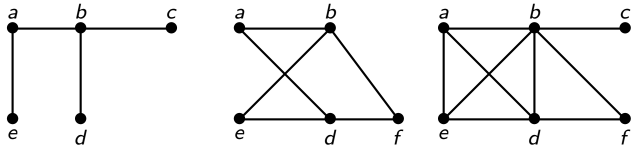
Graph Terminology

	Type	Edges	Multi-Edges	Loops
1	(Simple undirected) graph	Undirected	No	No
2	(Undirected) multigraph	Undirected	Yes	No
3	(Undirected) pseudograph	Undirected	Yes	Yes
4	Directed graph	Directed	No	Yes
5	Simple directed graph	Directed	No	No
6	Directed multigraph	Directed	Yes	Yes
8	Mixed graph	Both	Yes	Yes

Definition

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example 1 :



G_1 G_2 $G_1 \cup G_2$

Remark

The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an infinite graph, and in comparison, a graph with a finite vertex set and a finite edge set is called a finite graph.

In this course we will consider only finite graphs.

Definition

Two vertices a, b in a graph G are called adjacent (متجَاورة) in G if $\{a, b\}$ is an edge of G. If $e = \{a, b\}$ is an edge of G, then e is called incident with the vertices a and b or e connects a and b.

Definition

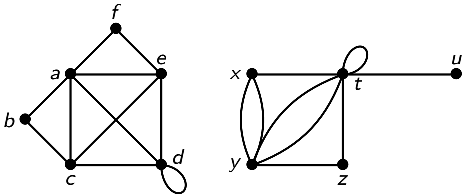
The degree of a vertex a in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex a is denoted by $\deg(a)$.

Definition

The neighborhood (neighbor set) of a vertex a in an undirected graph, denoted $N(a)$ is the set of vertices adjacent to a.

Example

Let F and G be the following graphs:



The degrees of the vertices in the graphs F and G are respectively:
$\deg(a) = 5$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 5$, $\deg(e) = 4$, $\deg(f) = 2$.
$\deg(x) = 3$, $\deg(y) = 5$, $\deg(z) = 2$, $\deg(t) = 7$, $\deg(u) = 1$.

$N(a) = \{b, c, d, e, f\}$, $N(b) = \{a, c\}$, $N(c) = \{a, b, d, e\}$.
$N(d) = \{a, c, e\}$, $N(e) = \{a, c, d, f\}$, $N(f) = \{a, e\}$.
$N(x) = \{y, z, t\}$, $N(y) = \{x, z, t\}$, $N(z) = \{x, y, t\}$,
$N(t) = \{x, y, z, t, u\}$, $N(u) = \{t\}$.

Definition

For any graph G,we define

$$\delta(G) = \min\{\deg v;\ v \in V(G)\}$$

and

$$\Delta(G) = \max\{\deg v;\ v \in V(G)\}.$$

If all the points of G have the same degree r, then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r.
A regular graph of degree 3 is called a cubic graph.

Handshaking Theorem

Theorem

If $G = (V, E)$ is a undirected graph with m edges, then:

$$2m = \sum_{a \in V} \deg(a).$$

Proof

Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

Corollary

Every cubic graph has an even number of points.

Proof

Let G be a cubic graph with p points, then $\sum_{v \in V} \deg(v) = 3p$ which is even by Handshaking Theorem. Hence p is even.

Corollary

An undirected graph has an even number of vertices of odd degree.

Proof

Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{a \in V_1} \deg(a) + \sum_{a \in V_2} \deg(a).$$

$\sum_{a \in V_1} \deg(a)$ must be even since $\deg(a)$ is even for each $a \in V_1$.
$\sum_{a \in V_2} \deg(a)$ must be even because $2m$ and $\sum_{a \in V_1} \deg(a)$ are even.

Example

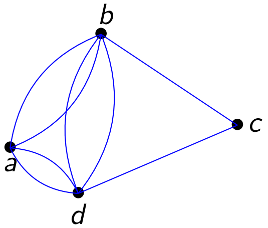
Every graph has with at least two vertices contains two vertices of equal degree.
Suppose that the all n vertices have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be $\{0, 1, 2, \ldots, n-2, n-1\}$.
But that's not possible, because the vertex with degree $n-1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

Example

If a graph has 7 vertices and each vertices have degree 6. The nombre of edges in the graph is 21. $(6 \times 7 = 42 = 2m = 2 \times 21)$.

Example

There is a graph with four vertices a, b, c, and d with $\deg(a) = 4$, $\deg(b) = 5 = \deg(d)$, and $\deg(c) = 2$.
The sum of the degrees is $4 + 5 + 2 + 5 = 16$. Since the sum is even, there might be such a graph with $\frac{16}{2} = 8$ edges.



Example

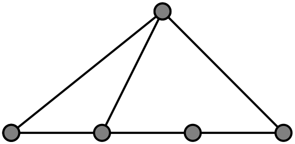
A graph with 4 vertices of degrees $1, 2, 3$, and 3 does not exist because $1 + 2 + 3 + 3 = 9$ (The Handshake Theorem.)
Also there is not a such graph because, there is an odd number of vertices of odd degree.

Example

For each of the following sequences, find out if there is any graph of order 5 such that the degrees of its vertices are given by that sequence. If so, give an example.

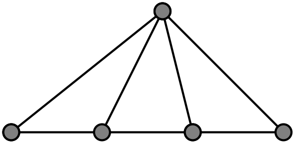
1. $3, 3, 2, 2, 2$
2. $4, 4, 3, 2, 1.$
3. $4, 3, 3, 2, 2.$
4. $3, 3, 3, 2, 2.$
5. $3, 3, 3, 3, 2.$
6. $5, 3, 2, 2, 2.$

1. $3, 3, 2, 2, 2$



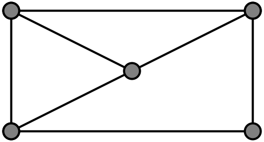
2. $4, 4, 3, 2, 1$. It does not exist. (One vertice v_1 which has degree 4, then there is one edge between v_1 and the others vertices. Also there is an other vertice v_2 which has degree 4, then there is one edge between v_2 and the others vertices. Then the minimum of degree is 2 and not 1).

3. $4, 3, 3, 2, 2.$



4. It does not exist. (The number of vertives with odd edges is odd).

5. $3, 3, 3, 3, 2.$

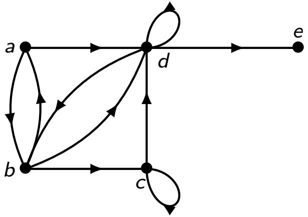


6. $5, 3, 2, 2, 2$. It does not exist. (The order is 5 and one vertive has degree 5).

Definition

The in-degree of a vertex a, denoted $\deg^-(a)$, is the number of edges directed into a. The out-degree of a, denoted $\deg^+(a)$, is the number of edges directed out of a. Note that a loop at a vertex contributes 1 to both in-degree and out-degree.

Example



In the graph we have: $\deg^-(a) = 1$, $\deg^+(a) = 2$, $\deg^-(b) = 2$, $\deg^+(b) = 3$, $\deg^-(c) = 2$, $\deg^+(c) = 2$, $\deg^-(d) = 4$, $\deg^+(d) = 3$, $\deg^-(e) = 1$, $\deg^+(e) = 0$.

Theorem

Let $G = (V, E)$ be a directed graph. Then:

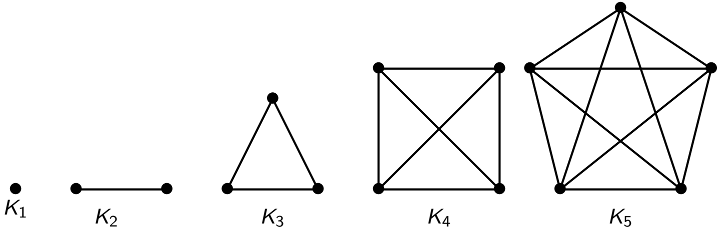
$$|E| = \sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v).$$

Proof The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. Both sums must be $|E|$.

Definition

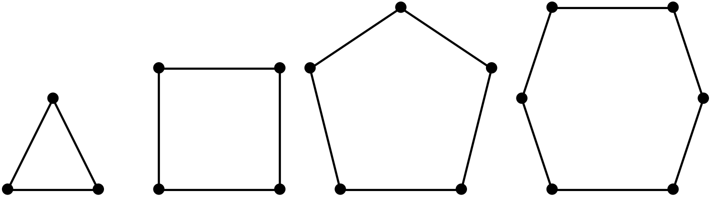
A null graph (or totally disconnected graph) is one whose edge set is empty. (A null graph is just a collection of points.)

A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



Cycles

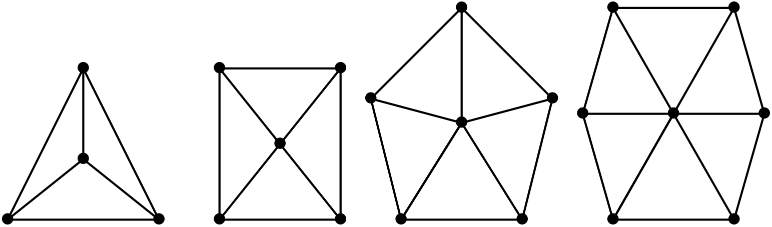
A cycle for $n \geq 3$ consists of n vertices v_1, v_2,..., v_n, and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,..., $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.



C_3 C_4 C_5 C_6

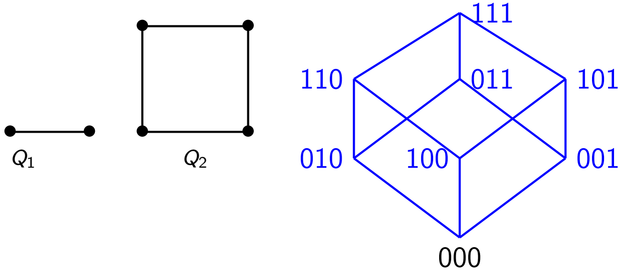
The Wheel Graph

The wheel graph W_n ($n \geq 3$) is obtained from C_n by adding a vertex a inside C_n and connecting it to every vertex in C_n.



W_3 W_4 W_5 W_6

An $n-$dimensional hypercube, or $n-$cube, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices if and only if they differ in exactly one bit position.



Bipartite Graphs

Definition

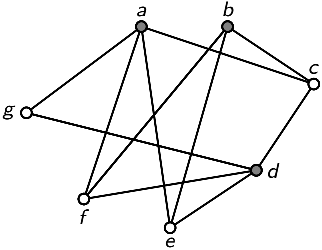
A bipartite graph is an (undirected) graph $G = (V, E)$ whose vertices can be partitioned into two disjoint sets (V_1, V_2), with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, such that for every edge $e \in E$, $e = \{a, b\}$ such that $a \in V_1$ and $b \in V_2$.

In other words, every edge connects a vertex in V_1 with a vertex in V_2. Equivalently, a graph is bipartite if and only if it is possible to color each vertex red or blue such that no two adjacent vertices have the same color.

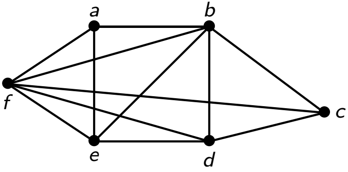
Bipartite Graphs

Definition

An equivalent definition of a bipartite graph is one where it is possible to color the vertices either red or blue so that no two adjacent vertices are the same color.



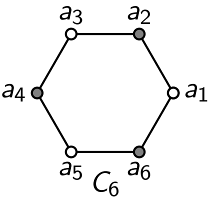
F

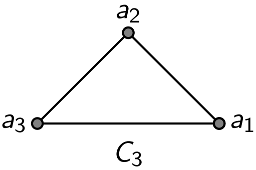


G

F is bipartite. $V_1 = \{a, b, d\}$, $V_2 = \{c, e, f, g\}$.
In G if we color a red, then its neighbors f and b must be blue.
But f and b are adjacent. G is not bipartite

Example





C_6 is bipartite. Partition the vertex set of C_6 into $V_1 = \{a_1, a_3, a_5\}$ and $V_2 = \{a_2, a_4, a_6\}$.
If we partition vertices of C_3 into two nonempty sets, one set must contains two vertices. But every vertex is connected to every other. So, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Theorem

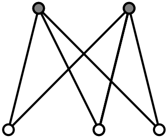
Let G be a graph of n vertices. Then G is bipartite if and only if it contains no cycles of odd length.

Complete Bipartite Graphs

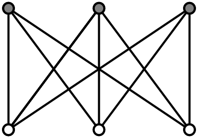
Definition

A complete bipartite graph is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

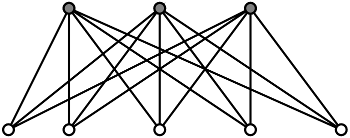
Example



$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

Subgraphs

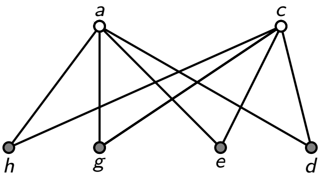
Definition

A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subset V$ and $F \subset E$.
A subgraph F of G is a proper subgraph of G if $F \neq G$.

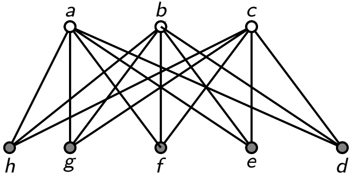
Induced Subgraphs

Definition

Let $G = (V, E)$ be a graph. The subgraph induced by a subset W of the vertex set V is the graph $H = (W, F)$, whose edge set F contains an edge in E if and only if both endpoints are in W.



$F = K_{2,4}$



$K_{3,5}$

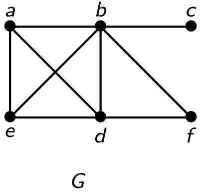
$K_{2,4}$ is the subgraph of $K_{3,5}$ induced by $W = \{a, c, e, g, h\}$.

Representing Graphs: Adjacency Lists

Definition

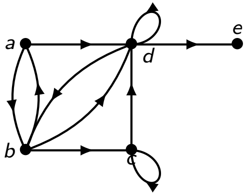
An adjacency list represents a graph (with no multiple edges) by specifying the vertices that are adjacent to each vertex.

Example



An adjacency list for a simply graph	
Vertex	Adjacent vertices
a	b, d, e
b	a, c, e, d, f
c	b
d	a, b, e, f
e	a, b, d
f	b, d

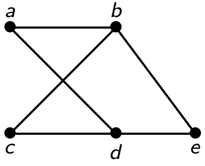
Example



An adjacency list for a directed graph	
Initial vertex	Terminal vertices
a	b, d
b	a, c, d
c	c, d
d	b, d, e
e	

Definition

Let $G = (V, E)$ be a simple graph where $|V| = n$. If $a_1, a_2, \ldots, a_n$ are the vertices of G. The adjacency matrix, A, of G, with respect to this listing of vertices, is the $n \times n$ matrix with its $(i, j)^{\mathrm{th}}$ entry is 1 if a_i and a_j are adjacent, and 0 if they are not adjacent.
($A = (a_{i,j})$, with $a_{i,j} = 1$ if $\{a_i, a_j\} \in E$ and $a_{i,j} = 0$ if $\{a_i, a_j\} \notin E$.)



G

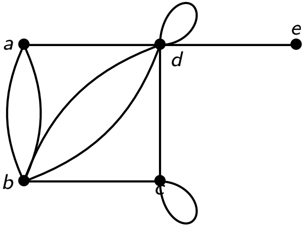
The adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The adjacency matrix of an undirected graph is symmetric: Also, since there are no loops, each diagonal entry is zero:

Example

The adjacency matrix for the following pseudograph is:



$$\begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

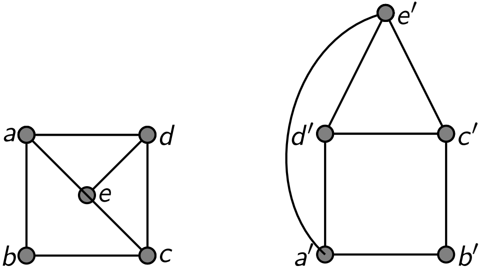
Definition

Two (undirected) graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic if there is a bijection, $f\colon V_1 \longrightarrow V_+2$, with the property that for all vertices $a, b \in V_1$

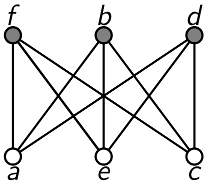
$$\{a, b\} \in E_1 \iff \{f(a), f(b)\} \in E_2.$$

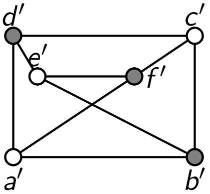
Such a function f is called an isomorphism.

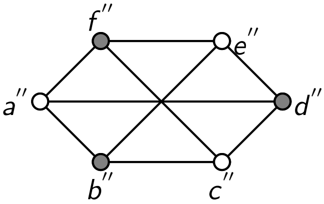
The following graphs are isomorphic.



The following graphs are isomorphic.







Theorem

Let f be an isomorphism of the graph $G_1 = (V_1, E_1)$ to the graph $G_2 = (V_2, E_2)$. Let $v \in V_1$. Then $\deg(v) = \deg(f(v)$. i.e., isomorphism preserves the degree of vertices.

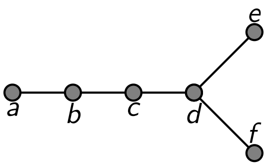
Proof A point $u \in V_1$ is adjacent to v in G_1 if and only if $f(u)$ is adjacent to $f(v)$ in G_2. Also f is bijection. Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to $f(v)$. Hence $\deg(v) = \deg(f(v))$.

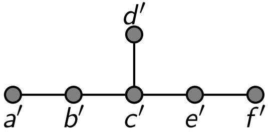
Remarks

1. Two isomorphic graphs have the same number of points and the same number of edges.
2. Two isomorphic graphs have equal number of points with a given degree.
   However these conditions are not sufficient to ensure that two graphs are isomorphic.

Example

Consider the two graphs given in figure below. Under any isomorphism d must correspond to c', a, e, f must correspond to a', d', f' in some order. The remaining two points b, c are adjacent whereas b', e' are not adjacent. Hence there does not exist an isomorphism.





Paths (in undirected graphs)

Informally, a path is a sequence of edges connecting vertices.

Definition

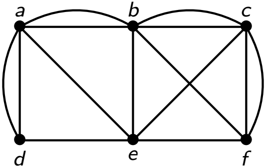
1. For an undirected graph $G = (V, E)$, an integer $n \geq 0$, and vertices $a, b \in V$, a path of length n from a to b in G is a sequence: $x_0, e_1, x_1, e_2, \ldots, x_{n-1}, e_n, x_n$ of interleaved vertices $x_j \in V$ and edges $e_i \in E$, such that $x_0 = a$ and $x_n = b$, and such that $e_i = \{x_{i-1}, x_i\} \in E$ for all $i \in \{1, \ldots, n\}$.
   Such path starts at a and ends at b.
   The trivial path from v to v consists of the single vertex v.

Definition

2. A path of length $n \geq 1$ is called a circuit (or cycle) if $n \geq 1$ and the path starts and ends at the same vertex, i.e., $a = b$.
3. A path or circuit is called simple if it does not contain the same edge more than once.

Remarks

1. When $G = (V, E)$ is a simple undirected graph a path $x_0, e_1, \ldots, e_n, x_n$ is determined uniquely by the sequence of vertices $x_0, x_1, \ldots, x_n$. So, for simple undirected graphs we can denote a path by its sequence of vertices $x_0, x_1, \ldots, x_n$.
2. Don't confuse a simple undirected graph with a simple path. There can be a simple path in a non-simple graph, and a non-simple path in a simple graph.



1. d, a, b, c, f is a simple path of length 4.
2. d, e, c, b, a, d is a simple circuit of length 5.
3. d, a, b, c, f, b, a, e is a path, but it is not a simple path, because the edge $\{a, b\}$ occurs twice in it.
4. c, e, a, d, e, f is a simple path, but it is not a tidy path, because vertex e occurs twice in it.

Paths in directed graphs

Definition

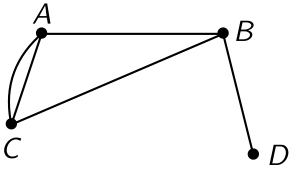
1. For a directed graph $G = (V, E)$, an integer $n \geq 0$, and vertices $a, b \in V$, a path of length n from a to b in G is a sequence of vertices and edges $x_0, e_1, x_1, e_2, \ldots, x_n, e_n$, such that $x_0 = a$ and $x_n = b$, and such that $e_i = (x_{i-1}, x_i) \in E$ for all $i \in \{1, \ldots, n\}$.
2. When there are no multi-edges in the directed graph G, the path can be denoted (uniquely) by its vertex sequence $x_0, x_1, \ldots, x_n$.
3. A path of length $n \geq 1$ is called a circuit (or cycle) if the path starts and ends at the same vertex, i.e., $a = b$.

Definition

4. A path or circuit is called simple if it does not contain the same edge more than once. (And we call it tidy if it does not contain the same vertex more than once, except possibly the first and last in case $a = b$ and the path is a circuit (cycle).)

Definition

An undirected graph $G = (V, E)$ is called connected, if there is a path between every pair of distinct vertices. It is called disconnected otherwise.



This graph is connected

Theorem

A graph G is connected if and only if for any partition of V into subsets V_1 and V_2 there is an edge joining a vertex of V_1 to a vertex of V_2.

Theorem

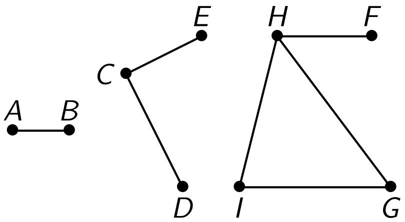
There is always a simple, and tidy, path between any pair of vertices a, b of a connected undirected graph G.

Proof By definition of connectedness, for every pair of vertices a, b, there must exist a shortest path $x_0, e_1, x_1, \ldots, e_n, x_n$ in G such that $x_0 = a$ and $x_n = b$. Suppose this path is not tidy, and $n \geq 1$. (If $n = 0$, the Proposition is trivial.) Then $x_j = x_k$ for some $0 \leq j < k \leq n$. But then $x_0, e_1, x_1, \ldots, x_j, e_{k+1}, x_{k+1}, \ldots, e_n, x_n$ is a shorter path from a to b, contradicting the assumption that the original path was shortest.

Connected Components of Undirected Graphs

Definition

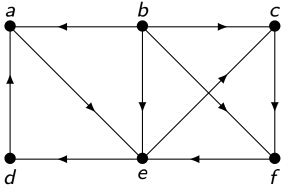
A connected component $H = (V', E')$ of a graph $G = (V, E)$ is a maximal connected subgraph of G, meaning H is connected and $V' \subset V$ and $E' \subset E$, but H is not a proper subgraph of a larger connected subgraph R of G.



This graph, $G = (V, E)$, has 3 connected components. (It is thus a disconnected graph.)

Definition

1. A directed graph $G = (V, E)$ is called strongly connected, if for every pair of vertices a and b in V, there is a (directed) path from a to b, and a directed path from b to a.
2. ($G = (V, E)$ is weakly connected if there is a path between every pair of vertices in V in the underlying undirected graph (meaning when we ignore the direction of edges in E.) A strongly connected component of a directed graph G, is a maximal strongly connected subgraph H of G which is not contained in a larger strongly connected subgraph of G.

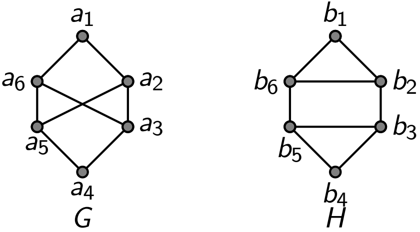


This digraph, G, is not strongly connected, because, for example, there is no directed path from e to b.

One strongly connected component of G is $H = (V_1, E_1)$, where $V_1 = \{a, c, d, e, f\}$ and $E_1 = \{(a, e), (e, c), (c, f), (f, e), (e, d), (d, a)\}$.

There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition, paths can be used to construct mappings that may be isomorphisms. As we mentioned, a useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2.

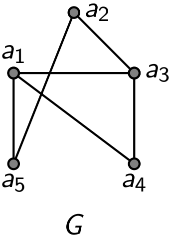
Let G and H be the following graphs.

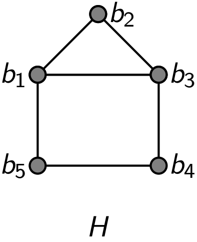


Both G and H have six vertices and eight edges. Each has 4 vertices of degree 3, and two vertices of degree 2. So, the three invariants number of vertices, number of edges, and degrees of vertices all agree for the two graphs. However, H has a simple circuit of length 3, namely, b_1, b_2, b_6, b_1, whereas G has no simple circuit of length 3. Then G and H are not isomorphic.

Example

Let G and H be the following graphs.

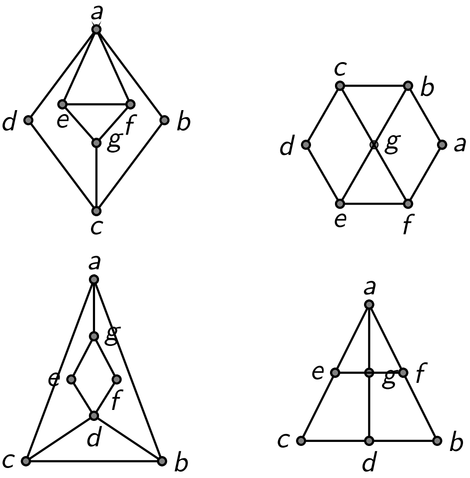




Both G and H have 5 vertices and 6 edges, both have 2 vertices of degree 3 and 3 vertices of degree 2, and both have a simple circuit of length 3, a simple circuit of length 4, and a simple circuit of length 5.

Because all these isomorphic invariants agree, G and H may be isomorphic.
To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths a_1, a_4, a_3, a_2, a_5 in G and b_3, b_2, b_1, b_5, b_4 in H both go through every vertex in the graph, start at a vertex of degree 3, go through vertices of degrees 2, three, and two, respectively, and end at a vertex of degree 2. By following these paths through the graphs, we define the mapping f with $f(a_1) = b_3$, $f(a_4) = b_2$, $f(a_3) = b_1$, $f(a_2) = b_5$, and $f(a_5) = b_4$.

Determine which of the graphs are isomorphic.



Counting Paths Between Vertices

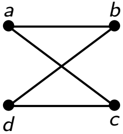
The number of paths between two vertices in a graph can be determined using its adjacency matrix.

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering $b_1, b_2, \ldots, b_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from b_i to b_j, where r is a positive integer, equals the $(i, j)^{th}$ entry of A^r.

Example

How many paths of length four are there from a to d in the simple graph G



The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Hence, the number of paths of length 4 from a to d is the $(1,4)^{th}$ entry of A^4. Because $A = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$.

There are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d.