Elementary Row Operations on Matrices

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Matrix and Matrix Operations

Definition

A real matrix is a rectangular array whose entries are real numbers. These numbers are organized on rows and columns. An $m \times n$ matrix will refer to one which has m rows and n columns, and the collection of all $m \times n$ matrices of real numbers will be denoted by $M_{m,n}(\mathbb{R})$. We adopt the notation, in which the $(j,k)^{th}$ entry of the matrix A (that in row j and column k) is denoted by $a_{j,k}$ and the matrix $A = (a_{j,k})$.
A matrix in $M_{m,n}(\mathbb{R})$ is called a matrix of dimension (or of type) (m,n).

Definition

• Two matrices $A = (a_{j,k})$ and $B = (b_{j,k})$ in $M_{m,n}(\mathbb{R})$ are called equal if $a_{j,k} = b_{j,k}$ for all j, k

• A matrix in $M_{1,n}(\mathbb{R})$ is called a row matrix.

• A matrix in $M_{m,1}(\mathbb{R})$ is called a column matrix

• If the entries of a matrix are zero, we denote this matrix (0) or 0

• A matrix in $M_{n,n}(\mathbb{R})$ is called a square matrix of type n and $M_{n,n}(\mathbb{R})$ will be denoted by $M_n(\mathbb{R})$

• A square matrix $A = (a_{j,k}) \in M_{n,n}(\mathbb{R})$ is called diagonal if $a_{j,k} = 0$ if $j \neq k$, example $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Definition

A square matrix $A = (a_{j,k}) \in M_{n,n}(\mathbb{R})$ is called upper triangular if $a_{j,k} = 0$ if $j > k$
A square matrix $A = (a_{j,k}) \in M_{n,n}(\mathbb{R})$ is called lower triangular if $a_{j,k} = 0$ if $j < k$
A diagonal matrix $A = (a_{j,k})$ in $M_n(\mathbb{R})$, where $a_{j,j} = 1$ is called the identity matrix and denoted by I_n.

Matrix Operations

Matrix algebra uses three different types of operations.

1. Matrix Addition: If $A = (a_{j,k})$ and $B = (b_{j,k})$ have the same dimensions (or the same type), then the sum $A + B$ is given by $A + B = (a_{j,k} + b_{j,k})$.
2. Scalar Multiplication: If $A = (a_{j,k})$ is a matrix and α a scalar (real number), the scalar product of α with A is given by $\alpha A = (\alpha a_{j,k})$.

3. Matrix Multiplication:
   1. If $A \in M_{1,n}(\mathbb{R})$ is a row matrix, $A = (a_1, \dots, a_n)$ and $B \in M_{n,1}(\mathbb{R})$ a column matrix, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, we define the product $A.B$ by:
   $$AB = a_1 b_1 + \cdots + a_n b_n.$$
   This matrix is of type $(1, 1)$ (one column and one row) and called the inner product of A and B.
   2. If $A = (a_{j,k}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{j,k}) \in M_{n,p}(\mathbb{R})$, then the product AB is defined as $AB = (c_{j,k}) \in M_{m,p}(\mathbb{R})$, where $c_{j,k}$ is the inner product of the j^{th} row of A with the k^{th} column of B
   $$c_{j,k} = \sum_{\ell=1}^{n} a_{j,\ell} b_{\ell k}.$$

The operations for matrix satisfy the following properties

Theorem

Let A, B, C *denote matrices in* $M_{m,n}(\mathbb{R})$, *and* $a, b \in \mathbb{R}$.

1. $A + B = B + A$,
2. $A + (B + C) = (A + B) + C$,
3. $a(A + B) = aA + aB$,
4. $(a + b)A = aA + bA$,
5. $(ab)A = a(bA)$,
6. *If* 0 *is the null matrix in* $M_{m,n}(\mathbb{R})$, *then* $A + 0 = A$.
7. $I_m A = A$ *and* $AI_n = A$,
   If $D \in M_{n,p}(\mathbb{R})$, $E \in M_{p,q}(\mathbb{R})$ *and* $F \in M_{r,m}(\mathbb{R})$, *then*
8. $A(DE) = (AD)E$,
9. $(A + B)D = AD + BD$,
10. $F(A + B) = FA + FB$,

Remarks

1. *The multiplication operation of matrix is not commutative i.e.* $AB \neq BA$ *in general. For example* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *Then* $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
2. *If* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *then* $A^2 = 0$.

Definition

The transpose of the matrix $A = (a_{j,k})$ in $M_{m,n}(\mathbb{R})$ is the matrix in $M_{n,m}(\mathbb{R})$, denoted by A^T and defined as follows:
$A^T = (b_{j,k})$, *where* $b_{j,k} = a_{k,j}$.

Theorem

Let $A, B \in M_{m,n}(\mathbb{R})$ and $C \in M_{n,p}(\mathbb{R})$, then

1. $(A + B)^T = A^T + B^T$,
2. $(AC)^T = C^T A^T$,
3. $\left(A^T\right)^T = A$.

Definition

A square matrix A is called symmetric if $A^T = A$.

Definition (The Elementary Row Operations)

There are three kinds of elementary matrix row operations:

1. *(Interchange) Interchange two rows,*
2. *(Scaling) Multiply a row by a non-zero constant,*
3. *(Replacement) Replace a row by the sum of the same row and a multiple of different row.*

Definition

Two matrix A and B in $M_{m,n}(\mathbb{R})$ are called row equivalent if B is the result of finite row operations applied to A. We denote $A \sim B$ if A and B are row equivalent. ($A \sim B$ is equivalent to $B \sim A$).

We denote the row operations as follows:

1. *The switches of the j^{th} and the k^{th} rows is indicated by: $R_{j,k}$*
2. *The multiplication of the j^{th} row by $r \neq 0$ is indicated by: $r \cdot R_j$.*
3. *The addition of r times the j^{th} row to the k^{th} row is indicated by: $rR_{j,k}$.*

Definition (Row Echelon Form)

A matrix in $M_{m,n}(\mathbb{R})$ is called in row echelon form if it has the following properties:

1. *The first non-zero element of a nonzero row must be 1 and is called the leading entry.*
2. *All non-zero rows are above any rows of all zeros,*
3. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*

Definition (Reduced Echelon Form)

A matrix in $M_{m,n}(\mathbb{R})$ is called in reduced row echelon form if it has the following properties:

1. *The matrix is in row echelon form,*
2. *Each leading number is the only non-zero entry in its column.*

Example

1. $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ is in row echelon form but is not reduced:
2. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ is in reduced row echelon form:
3. $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 5 \\ 3 & 0 & 0 \end{pmatrix}$ is not in row echelon form.

Example

$$\begin{pmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 4 & 1 & 0 \end{pmatrix} \xrightarrow{-1R_{1,2}} \begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_{1,2}} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 4 & 1 & 0 \end{pmatrix} \xrightarrow{-2R_{1,2},-4R_{1,3}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \\ 0 & 9 & -12 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 9 & -12 \end{pmatrix} \xrightarrow{-9R_{2,3}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_3} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-3R_{3,1},1.R_{3,2}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_{2,1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 2 & -3 & 4 & -2 & 0 \\ 3 & -1 & 2 & -3 & 2 \\ -2 & -3 & 4 & 3 & 2 \\ -3 & 1 & 0 & 3 & 1 \end{pmatrix} \xrightarrow{(-1)R_{1,2}} \begin{pmatrix} 2 & -3 & 4 & -2 & 0 \\ 1 & 2 & -2 & -1 & 2 \\ -2 & -3 & 4 & 3 & 2 \\ -3 & 1 & 0 & 3 & 1 \end{pmatrix}$$

$$\xrightarrow{R_{1,2}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 2 & -3 & 4 & -2 & 0 \\ -2 & -3 & 4 & 3 & 2 \\ -3 & 1 & 0 & 3 & 1 \end{pmatrix} \xrightarrow{(-2)R_{1,2}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & -7 & 8 & 0 & -4 \\ -2 & -3 & 4 & 3 & 2 \\ -3 & 1 & 0 & 3 & 1 \end{pmatrix}$$

$$\xrightarrow{2R_{1,3},3R_{1,4}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & -7 & 8 & 0 & -4 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 7 & -6 & 0 & 7 \end{pmatrix} \xrightarrow{R_{2,3},1R_{2,4}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & -7 & 8 & 0 & -4 \\ 0 & 0 & 2 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{7R_{2,3}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 8 & 7 & 38 \\ 0 & 0 & 2 & 0 & 3 \end{pmatrix} \xrightarrow{R_{3,4}} \begin{pmatrix} 1 & 2 & -2 & -7 & 2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 8 & 7 & 38 \end{pmatrix}$$

$$\xrightarrow{-4R_{3,4}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 7 & 26 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3, \frac{1}{7}R_4} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{26}{7} \end{pmatrix}$$

$$\xrightarrow{-2R_{2,1}} \begin{pmatrix} 1 & 0 & -2 & -3 & -10 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{26}{7} \end{pmatrix} \xrightarrow{2R_{3,1}} \begin{pmatrix} 1 & 0 & 0 & -3 & -7 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{26}{7} \end{pmatrix}$$

$$\xrightarrow{3R_{4,1}} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{29}{7} \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{26}{7} \end{pmatrix} \xrightarrow{(-1)R_{4,2}} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{29}{7} \\ 0 & 1 & 0 & 0 & \frac{16}{7} \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{26}{7} \end{pmatrix}$$

Fractions can be avoided as follows:

$$\xrightarrow{-4R_{3,4}} \begin{pmatrix} 1 & 2 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 7 & 26 \end{pmatrix} \xrightarrow{7R_1, 7R_2} \begin{pmatrix} 7 & 14 & -14 & -7 & 14 \\ 0 & 7 & 0 & 7 & 42 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 7 & 26 \end{pmatrix}$$

$$\xrightarrow{1R_{4,1}, -1R_{4,2}} \begin{pmatrix} 7 & 14 & -14 & 0 & 40 \\ 0 & 7 & 0 & 0 & 16 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 7 & 26 \end{pmatrix} \xrightarrow{7R_{3,1}, -2R_{2,1}} \begin{pmatrix} 7 & 0 & 0 & 0 & 29 \\ 0 & 7 & 0 & 0 & 16 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 7 & 26 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{7}R_1, \frac{1}{7}R_2, \frac{1}{2}R_3, \frac{1}{7}R_4} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{29}{7} \\ 0 & 1 & 0 & 0 & \frac{16}{7} \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{26}{7} \end{pmatrix}$$

Theorem

Each matrix is row equivalent to one and only one reduced echelon matrix.

Definition

We say that a square matrix A of type (n, n) (or of order n) is invertible if there is a square matrix B of type (n, n) such that $AB = BA = I_n$.
We denote A^{-1} the inverse matrix of A.

Theorem

A matrix A is invertible if there is a square matrix B such that $AB = I_n$.
The inverse matrix of a matrix A is unique and will be denoted by A^{-1}.

Theorem

1. The inverse matrix if it exists is unique,
2. The inverse matrix of I_n is I_n.
3. $(A^{-1})^{-1} = A$.
4. If A and B have inverses, then $(AB)^{-1} = B^{-1}A^{-1}$.
5. If $A_1, \dots, A_k$ are invertible, then

$$(A_1.\dots.A_k)^{-1} = A_k^{-1}.\dots.A_1^{-1}.$$

6. If A is invertible, then $(rA)^{-1} = \frac{1}{r}A^{-1}$, for all $r \in \mathbb{R}^*$.
7. If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Definition

We say that a matrix E of order n is an elementary matrix if it is the result of applying a row operation to the identity matrix I_n.

Remarks

1. Let the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 1 & -2 & 1 \end{pmatrix}$ and the elementary matrix $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ which is the result of switching the second and the third rows of I_3.

   We have $R_{2,3}A = EA = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$.

2. An other example: let $A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$ and the elementary matrix $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} = 5R_{1,3}I_3$.

   We have $5R_{1,3}A = EA = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 6 & 4 & 14 & 15 \end{pmatrix}$.

In general we have

Theorem

For all $A \in M_{m,n}(\mathbb{R})$ and R an elementary row operation on $M_{m,n}(\mathbb{R})$, E an elementary matrix such that $E = R(I_m)$. Then

$$EA = R(A)$$

where $R(A)$ is the result of the elementary row operation R on A.

Theorem

If E is an elementary matrix, then E has an inverse and its inverse is an elementary matrix.

Theorem

If A is a square matrix of order n. The following are equivalent:

1. The matrix A has an inverse.
2. The reduced row echelon form of the matrix A is I_n.
3. There is a finite number of elementary matrices $E_1, \ldots, E_m$ in $M_n(\mathbb{R})$ such that $A = E_1 \ldots E_m$.

(Algorithm)

Let $A \in M_n(\mathbb{R})$

1. Let $[B|C]$ be the reduced row echelon form of the matrix $[A|I] \in M_{n,2n}(\mathbb{R})$.
2. If $B = I_n$, then $C = A^{-1}$.
3. If $B \neq I_n$, the matrix A is not invertible.

Example

The inverse matrix of the matrix $A = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_{1,2}, R_{2,3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_{1,2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)R_{2,3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & 0 & \frac{1}{2} & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{2R_2, 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right]$$

$$\xrightarrow{(-1)R_{3,1}, 2R_{3,2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right]$$

$$\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

The inverse matrix of the matrix $A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 3 & 3 & 1 \\ 3 & 3 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 1 & 0 & 0 \\ 3 & 3 & 4 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_{1,4}]{(-2)R_{1,2},(-3)R_{1,3}} \left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -6 & -2 & -1 & -3 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_4]{(-1)R_2,(-1)R_3} \left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 12 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 2 & -1 & 0 & 0 \\ 0 & 6 & 2 & 1 & 3 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[(-2)R_{2,3}]{(-1)R_{4,2}} \left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{(-2)R_{2,4}} \left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 & 0 & -3 \end{array}\right]$$

$$\underset{(-2)R_{3,4}}{\xrightarrow{(1)R_{3,2},-1R_3}} \left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 2 & -3 \end{array}\right]$$

$$\xrightarrow{R_{3,4}} \left[\begin{array}{cccc|cccc} 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 2 & -3 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[(-1)R_{4,1}]{(-3)R_{2,1},(-2)R_{3,1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 3 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 2 & -3 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 \end{array}\right].$$

The inverse matrix of the matrix A, where $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 1 \end{array}\right]$$

Then $A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -2 & 1 & 1 & 1 \end{pmatrix}$.